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# Path Integrals and Perturbative Expansions for Non-Compact Symmetric Spaces

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**Abstract:** We show how to construct path integrals for quantum mechanical systems where the space of configurations is a general non-compact symmetric space. Associated with this path integral is a perturbation theory which respects the global structure of the system. This perturbation expansion is evaluated for a simple example and leads to a new exactly soluble model. This work is a step towards the construction of a strong coupling perturbation theory for quantum gravity.

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## 1. Introduction

The construction of a quantum theory of fundamental processes that is central to physics. Despite this, quantization of even the most elementary systems has proved to be far from straightforward. For a system whose configuration space is the whole of the real line, $\mathbf{R}$, or any finite dimensional vector space, there are a number of equivalent quantization schemes. One approach is the canonical operator formalism in which the phase space variables $x$ and $p$ are replaced by operators $\hat{x}$ and $\hat{p}$, and their Poisson bracket

$$\{x, p\} = 1 \tag{1.1}$$

is replaced by the commutator

$$[\hat{x}, \hat{p}] = i, \tag{1.2}$$

the Heisenberg-Weyl algebra. Quantization is completed by finding an irreducible unitary representation of this algebra; it may be shown that any such representation is unitarily equivalent to the usual one on the space of square-integrable complex-valued functions on $\mathbf{R}$, $L^2(\mathbf{R}, \mathbf{C})$ [1].

An alternative and very beautiful formulation of quantum mechanics on $\mathbf{R}$ is the path integral. In this approach, the transition amplitude between an initial state $i$ at time $t_i$, and a final state $f$ at time $t_f$, is

$$G(f, t_f; i, t_i) = \int \mathcal{D}[x(t), p(t)] e^{i \int_{t_i}^{t_f} dt(p\dot{x} - H(p,x))} \tag{1.3}$$

In the path integral, $\mathcal{D}[x(t), p(t)]$ is the Feynman-Wiener measure on the space of all paths in phase space that interpolate between the inital and final configurations. This formula applies to any system with Hamiltonian $H$. and can be shown to be equivalent to the operator formulation under certain circumstances [2].

In order to treat more complicated problems, we would like to extend the ideas of path integration to more general situations such as are encountered in condensed matter physics, gravitation and $\sigma$-models. Our intention is to construct a path integral whose perturbation theory respects the global structure of the system. In a previous publication [3], we constructed a quantum mechanical path integral for a system whose configuration space is $\mathbf{R}^+$, the positive real line, and applied it to a minisuperspace model of quantum gravity. We showed how the fact that the configuration space variable was restricted to be positive led to a different form of the path integral measure than is usually assumed, and that the unitarity of time evolution and the question of topology change are crucially affected by the correct identification of the path integral measure.

It seems clear that related considerations will have a considerable bearing on any complete quantum theory of gravity, and this paper is a step in that direction. Specifically, we extend the construction of the path integral in [3] to a more general finite-dimensional non-compact configuration space of the form $G/H$, where $G$ and $H$ are two Lie gorups. As we shall show, this path integral respects the global structure of the system and allows the development of a perturbation theory which goes beyond the conventional weak field expansion in analogous field theory systems.

Although we shall treat quantum mechanics in this paper, our motivation is the field theoretic analog to which we intend to return in a future paper. Our primary aim here is to develop a perturbation scheme which does not have the drawbacks of the Feynman diagram approach usually employed. The first drawback is that conventional perturbation theory is based on Gaussian integration, and this means that one allows the integration variables to range from minus infinity to infinity which is an approximation that does not respect the global structure of general non-linear systems. Secondly, for many non-compact configuration spaces, the normal coordinate expansion does not probe the whole space: for example the matrix $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in SL(2, \mathbf{R})$ cannot be written in the form $\exp(\alpha^i T^i)$ where $T^i$ are the generators of $SL(2, \mathbf{R})$. Thus the Vilkovisky-DeWitt approach, while reflecting the local geometry, does not allow one to probe the full range of quantum fluctuations.

A further motivation comes from quantum gravity, for which the space of configurations, positive definite symmetric matrices at each point in space modulo spatial diffeomorphisms, is an example of the field theoretic extension of this work. It is known that the weak field perturbation theory for this system is unsatisfactory because it is unrenormalizable, however the approach adopted here is naturally tailored to a strong field perturbation theory [4]. Although this is the most interesting regime of quantum gravity, it is the least understood and we expect the work described here to be a step towards allowing us to probe this domain.

We should make it clear that the path integral described here is an alternative to the usual configuration space or phase space path integrals which are also correct in the sense that they satisfy the relevant Schrödinger equation [2]. We argue, however, that the path integral we propose is useful if one wishes to do the sort of "global" perturbation theory we are interested in, whereas the configuration space path integral is more adapted to weak field perturbation theory .

The construction we shall present takes as its starting point the canonical quantum mechanics of the relevant system, which we shall take to be the finite dimensional coset space $G/H$ where $G$ is non-compact and $H$ is compact. As has been explained elsewhere [5], [6], the procedure for finding an algebra of functions on the classical phase space closed under Poisson brackets and then turning these functions into operators leads to the "canonical" group of the semi-direct product form, $W \tilde{\times} G$, where $W$ is a vector space in which $G$ acts linearly with an orbit $G/H$.

The space of quantum states corresponds to unitary representations of this group. There may be many inequivalent quantizations of this system, in contrast to the case of quantum mechanics on $\mathbf{R}$. The inequivalent quantizations of $G/H$ are labelled by irreducible representations $\pi_\chi$ of $H$, with the quantization corresponding to $\pi_\chi$ realised on the Hilbert space of square-integrable functions taking their values in the vector space $\mathcal{H}_\chi$ carrying the (finite-dimensional) irreducible representation $\pi_\chi$ of $H$, *ie.* the Hilbert space is $L^2(G/H, \mathcal{H}_\chi)$, [5], [6]. Amongst these quantisations is the "usual" one on $L^2(G/H, \mathbf{C})$, corresponding to the trivial representation of $H$.

As a first example, we shall concentrate on the case of $n \times n$ positive symmetric matrices, $Symm^+(n, \mathbf{R})$, that is symmetric matrices whose eigenvalues are all positive definite. This space is isomorphic to the coset space $GL(n, \mathbf{R})/O(n)$ [7] since $GL(n, \mathbf{R})$ acts transitively on $Symm^+(n, \mathbf{R})$ by conjugation,

$$Y \mapsto g^T Y g \quad Y \in Symm^+(n, \mathbf{R}), g \in GL(n, \mathbf{R}) \ , \tag{1.4}$$

with little group at the identity, $O(n)$. The canonical group [7] is $Symm(n, \mathbf{R}) \tilde{\times} GL(n, \mathbf{R})$ with Lie algebra relations

$$\begin{aligned}
&[\hat{Y}_{ab}, \hat{Y}_{cd}] = 0 \\
&[\hat{\pi}_a{}^b, \hat{\pi}_c{}^d] = i(\delta_a^d \hat{\pi}_c{}^b - \delta_c^b \hat{\pi}_a{}^d) \\
&[\hat{Y}_{ab}, \hat{\pi}_c{}^d] = i(\delta_a^d \hat{Y}_{bc} + \delta_b^d \hat{Y}_{ac}),
\end{aligned} \tag{1.5}$$

where $Symm(n, \mathbf{R})$ is space of all real $n \times n$ symmetric matrices, and is isomorphic to $\mathbf{R}^{\frac{1}{2}n(n+1)}$. The natural $GL(n, \mathbf{R})$ invariant Hamiltonian on $Symm^+(n, \mathbf{R})$ is

$$\hat{H}_0 = \frac{1}{2} \hat{\pi}_a{}^b \hat{\pi}_b{}^a \ . \tag{1.6}$$

In this paper, we shall treat only the quantisation in which $\mathcal{H}_\chi = \mathbf{C}$ and in this case the representation of $\hat{\pi}$ and $\hat{Y}$ is given by

$$\begin{aligned}
&\hat{\pi}_a{}^c \psi(Y) = -i Y_{ab} \left( \frac{\partial}{\partial Y} \right)_{bc} \psi(Y) \\
&\hat{Y}_{ab} \ \psi(Y) = Y_{ab} \ \psi(Y)
\end{aligned} \quad . \tag{1.7}$$

with

$$\left(\frac{\partial}{\partial Y}\right)_{ab} = (1+\delta_{ab})\left(\frac{\partial}{\partial Y_{ab}}\right) \quad (no \;\; sum) \tag{1.8}$$

The plan of this paper is as follows: in section 2 we outline how to carry out Fourier analysis on $Symm^+(n, \mathbf{R})$ which we then use to construct a formal path integral. In section 3 we shall show how this form of the path integral, rather than the configuration space or phase space form, leads naturally to the "global" perturbation theory; we then calculate the first order in perturbation theory for the quantum mechanics of a particle moving on $Symm^+(n, \mathbf{R})$ in a simple potential, in order to demonstrate that the terms in the perturbation expansion may be evaluated in detail. The form of this expansion suggests that this highly nonlinear model has an exact solution. We verify that this is indeed the case. In section four, we show how this construction can be generalized to more general non-compact symmetric spaces. Finally, some conclusions are presented in section 5.

## 2. Fourier-Helgason Transform and Path Integral

The Fourier modes $e^{ipx}$ on the real line $\mathbf{R}$ have two important properties. The first is that they are "complete" in the sense that the Fourier transform $\tilde{f}(p)$ of a function $f(x)$, given by

$$\tilde{f}(p) = \int_{-\infty}^{\infty} f(x)\; e^{ipx}\; dx, \tag{2.1}$$

may be inverted by the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(p)\; e^{-ipx}\; dp, \tag{2.2}$$

and secondly that they are eigenfunctions [1] of the invariant differential operators on $\mathbf{R}$, the most important example of which is the Laplacian $\frac{d^2}{dx^2}$.

These two properties are extremely useful since they allow one to construct the propagator for a free particle, that is the propagator for the Schrödinger equation with free Hamiltonian $-\frac{1}{2}\frac{d^2}{dx^2}$. It is

$$G_0(x_f, t_f; x_i, t_i) = \int_{-\infty}^{\infty} dp\; e^{ip(x_f - x_i)}\; e^{\frac{1}{2}ip^2(t_f - t_i)}\;. \tag{2.3}$$

[1] Since $e^{ipx}$ is not normalizable with respect to the usual $L^2$ inner product, it is not, strictly speaking an eigenfunction; this problem may, however, be treated by rigged Hilbert space techniques [8].

These Fourier modes are also used in the construction of the path integral for a particle moving on **R**. In order to find the phase space path integral expression for the propagator $\langle x_f, t_f \mid x_i, t_i\rangle$ one inserts complete sets of states $\mid x\rangle$ and $\mid p\rangle$ at $N$ intermediate intervals. $\mid x\rangle$ is an eigenstate of the operator $\hat{x}$ with eigenvalue $x$ and $\mid p\rangle$ is an eigenstate of $\hat{p}$ with eigenvalue $p$. They satisfy the completeness relations

$$\int_{-\infty}^{\infty} dx \mid x\rangle\langle x \mid = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \mid p\rangle\langle p \mid = 1 \ . \tag{2.4}$$

Thus

$$\begin{aligned}
&\langle x_f, t_f | x_i, t_i\rangle \\
&= \left(\prod_{r=1}^{N}\int_{-\infty}^{\infty} dx_r \left(\frac{1}{2\pi}\right)^N \int_{-\infty}^{\infty} dP_r\right) \ \langle x_f, t_f | x_N, t_N\rangle\langle x_N, t_N | P_N, t_N\rangle \ldots \\
&\ldots |x_{r+1}, t_{r+1}\rangle\langle x_{r+1}, t_{r+1} | P_{r+1}, t_{r+1}\rangle\langle P_{r+1}, t_{r+1} | x_r, t_r\rangle\langle x_r, t_r| \ldots \\
&\ldots \langle P_1, t_1 | x_i, t_i\rangle.
\end{aligned} \tag{2.5}$$

We may now use the fact that

$$\langle x_r, t_r | P_r, t_r\rangle = e^{iP_r x_r} \tag{2.6}$$

and

$$\langle P_{r+1}, t_{r+1} | x_r, t_r\rangle = e^{-iP_{r+1}x_r - i(t_{r+1}-t_r)H(x_r, P_{r+1})} \tag{2.7}$$

to order $(t_{r+1} - t_r)$ and hence implicitly to lowest order in $\hbar$.

Therefore

$$\begin{aligned}
\langle x_f, t_f | x_i, t_i\rangle \simeq \lim_{N\to\infty} &\left(\prod_{r=1}^{N}\int_{-\infty}^{\infty} dx_r \left(\frac{1}{2\pi}\right)^N \int_{-\infty}^{\infty} dP_r\right) \ \delta(x_f - x_N) \\
&\exp i\left\{\sum_{r=0}^{N-1}\Big(P_{r+1}(x_{r+1} - x_r) - \Delta t \ H(x_r, P_{r+1})\Big)\right\}
\end{aligned} \tag{2.8}$$

so that, were the limit to exist,

$$\langle x_f, t_f | x_i, t_i\rangle \text{ ` = ' } Z\int \mathcal{D}P \int \mathcal{D}x \exp\left\{i\int_{t_i}^{t_f} dt\,(P(t)\dot{x}(t) - H(x, P))\right\} \tag{2.9}$$

where $Z$ is a normalisation constant.

This last equation is the usual starting point for perturbative calculations. If $H = H_0+\lambda V$ with some Hamiltonian $H_0$ for which the path integral is known, and potential $\lambda V$, then by standard methods one can calculate the path integral perturbatively as a series in positive powers of $\lambda$. However, since the limit $N \to \infty$ does not really exist, one can argue that the path integral is largely a mnemonic and is correct in as much as it reproduces this expansion correctly. This is not to say that the path integral is devoid of content; indeed the path integral can be used to treat various phenomena non-perturbatively, an example is provided by the use of instantons to treat tunneling processes. The expansion of the propagator $G_V$ for $H = H_0 + \lambda V$ in positive powers of $\lambda$ is most easily derived from its integral equation [9], namely

$$G_V(x_f,t_f;x_i,t_i) = G_0(x_f,t_f;x_i,t_i) - \lambda \int_{t_i}^{t_f} dt \int_{-\infty}^{\infty} dx \;\; G_0(x_f,t_f;x,t)V(x)G_V(x,t;x_i,t_i)\ . \tag{2.10}$$

Thus the perturbation series may be written, schematically, as

$$G_V = G_0 + \lambda G_0 V G_0 + \lambda^2 G_0 V G_0 V G_0 + ...., \tag{2.11}$$

where

$$G_0 V G_0 = \int_{t_i}^{t_f} dt \int_{-\infty}^{\infty} dx G_0(x_f,t_f;x,t)V(x)G_0(x,t;x_i,t_i)\ , \tag{2.12}$$

and so forth for higher order terms.

We have set out this well-known derivation in detail since it may be repeated in much more general situations such as the general (non-trivially induced) quantisations of motion on the arbitrary finite-dimensional coset spaces $G/H$ in which we are interested.

Rather than treating the general case here, we shall start with the example of motion on the space of $n \times n$ positive symmetric matrices $Symm^+(n, \mathbf{R})$. The beginning of this section follows Terras [10] closely. The analogs of the Fourier modes in this case are the so-called power functions $p_{\mathbf{s}}(Y[k])$, where $Y \in Symm^+(n, \mathbf{R})$. These functions are labelled by $\mathbf{s} \in \mathbf{C}^n$, $k \in O(n)$ and $Y[k]$ means conjugated $Y$ by $k$, thus $Y[k] = k^T Y k$. Then

$$p_{\mathbf{s}}(Y) = \prod_{j=1}^{n} |det\ Y_j|^{s_j}\ , \tag{2.13}$$

where $Y_j$ is the $j \times j$ matrix obtained by taking the first $j$ rows and columns of $Y$. Power functions are eigenfunctions of the invariant differential operators $L$. For $GL(n, \mathbf{R})$, a basis for the commutative ring of invariant differential operators is given by

$$L^{(r)} = \mathrm{Tr}\ \pi^r \tag{2.14}$$

with $r = 1 \ldots n$. For any invariant differential operator $L$, the eigenvalues $\lambda_L$ depend only on $\mathbf{s}$

$$L\ p_{\mathbf{s}}(Y[k]) = \lambda_L(\mathbf{s})\ p_{\mathbf{s}}(Y[k])\ . \tag{2.15}$$

The Fourier-Helgason transform $\tilde{f}(\mathbf{s}, k)$, of a function $f(Y)$ on $Symm^+(n, \mathbf{R})$ is given by

$$\tilde{f}(\mathbf{s}, k) = \int_{Symm^+(n,\mathbf{R})} d\mu(Y)\ f(Y)\ \overline{p_{\mathbf{s}}(Y[k])} \tag{2.16}$$

where $d\mu(Y)$ is the measure on $Symm^+(n, \mathbf{R})$ invariant under $GL(n, \mathbf{R})$, and the bar indicates complex conjugation. The inversion formula is

$$f(Y) = \omega_n \int_{\mathbf{s}=\mathbf{v}+i\mathbf{u}} ds \int_{\bar{k} \in K/M} d\bar{k}\ \tilde{f}(\mathbf{s}, k)\ p_s(Y[k])\ |c_n(\mathbf{s})|^{-2} \tag{2.17}$$

where

$$\mathbf{v} = (\frac{1}{2}, \frac{1}{2}, \ldots.., \frac{1}{2}, \frac{1}{4}(1-n)) \tag{2.18}$$

and each component of $\mathbf{u}$ is real and runs from $-\infty$ to $\infty$.

$$\omega_n = \prod_{j=1}^{n} \frac{\Gamma(j/2)}{j(2\pi i)\pi^{j/2}} \tag{2.19}$$

and the Harish-Chandra $c$-function is given in terms of the beta-function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ , \tag{2.20}$$

as

$$c_n(\mathbf{s}) = \prod_{1 \le i \le j \le n-1} \frac{B(\frac{1}{2}, s_i + \ldots. + s_j + \frac{1}{2}(j-i+1))}{B(\frac{1}{2}, \frac{1}{2}(j-i+1))}\ . \tag{2.21}$$

Finally, the space over which the $k$'s are integrated is the coset space $K/M$ where $K$ is $O(n)$ and $M$ is the set of diagonal matrices with non-zero entries of $\pm 1$. That this integral is over $\bar{K}$ rather than $K$ arises because of the degeneracy in specifying $k$ given a particular power function coming from a specified $Y$. In other words $M$ is the automorphism group of the invariant power functions. By convention, one normalizes the $\bar{k}$ integration over $K/M$ to be unity

$$\int_{K/M} d\bar{k} = 1 \tag{2.22}$$

For simplicity we shall write the above inverse transform as

$$f(Y) = \int d\mu_{Helg}(\mathbf{s}, k)\ \tilde{f}(\mathbf{s}, k) p_{\mathbf{s}}(Y[k]) \tag{2.23}$$

As in the case of quantum mechanics on $\mathbf{R}$, we may use this transform to give an expression for the free propagator where the Hamiltonian is the invariant Laplacian on $Symm^{+}(n, \mathbf{R})$. The propagator $G_0(Y_f, t_f; Y_i, t_i)$ is the solution to

$$H_0 G_0(Y_f, t_f; Y_i, t_i) = i \frac{\partial}{\partial t_i} G_0(Y_f, t_f; Y_i, t_i), \tag{2.24}$$

the Schrödinger equation, with $H_0 = \frac{1}{2} \hat{\pi}_a{}^b \hat{\pi}_b{}^a$ and $\lim_{t_i \to t_f} G_0(Y_f, t_f; Y_i, t_i) = \delta(Y_f, Y_i)$, and it is given by

$$G_0(Y_f, t_f; Y_i, t_i) = \int d\mu_{Helg}(\mathbf{s}, k) p_{\mathbf{s}}(Y_i[k]) \overline{p_{\mathbf{s}}(Y_f[k])}\ e^{i\lambda_{H_0}(\mathbf{s})(t_f - t_i)}. \tag{2.25}$$

We may also use the power functions to form a path integral for this system just as in the case of quantum mechanics on $\mathbf{R}$. We insert complete sets of states,

$$\int d\mu_{inv}(Y)\ |Y\rangle\langle Y| = \int d\mu_{Helg}(\mathbf{s}, k)\ |\mathbf{s}, k\rangle\langle \mathbf{s}, k| = 1 \tag{2.26}$$

to produce the following expression for the propagator associated with the Hamiltonian $H_0 + \lambda V$

$$\begin{aligned} \langle Y_f, t_f | Y_i, t_i \rangle = \int \left( \prod_{n=1}^{N} d\mu_{Helg}(\mathbf{s}_n, k_n) d\mu_{inv}(Y_n) \right) e^{\sum_{n=1}^{N} \ln\left( p_{\mathbf{s}}(Y_n[k]) \overline{p_{\mathbf{s}}(Y_{n-1}[k])} \right)} \\ e^{i \sum_{n=1}^{N} (\lambda_{H_0}(\mathbf{s}_n) + \lambda V(Y_n)) \Delta t}. \end{aligned} \tag{2.27}$$

We note that this is not the usual phase-space path integral. However, expanding in powers of $\lambda$ gives the usual canonical perturbation expansion in which all integrals are over measures and ranges appropriate to the global structure of the problem under consideration.

## 3. The Perturbation Expansion

We have argued that if we want to develop a perturbation theory that respects the global structure of the space of configurations, we should use (2.27) rather than the usual phase space path integral, which is tailored to perturbation theory using Gaussian integrals and hence only respects the local geometry. The usual expression does, at least, have the benefit of allowing calculation of the perturbation series. We now show that the terms in the perturbation series based on (2.27) may also be calculated even though at first sight the relevant integrals look intractable.

For definiteness we consider the case of $2 \times 2$ positive symmetric matrices and the Hamiltonian $H_0 + \lambda \ln \det(Y)$, where $H_0$ is the free Hamiltonian, given by the invariant Laplacian. We now evaluate $G_V$ to the lowest order in perturbation theory in $\lambda$.

The first order term in the expansion is thus

$$\lambda \int_{t_i}^{t_f} dt \int d\mu_{inv}(Y)\ G_0(Y_f, t_f; Y, t)\ \ln\det(Y)\ G_0(Y, t; Y_i, t_i)\ . \tag{3.1}$$

We show that this is equal to

$$\left(\frac{1}{2}(t_f - t_i)\left(\ln\det(Y_i) + \ln\det(Y_f)\right)\right) G_0(Y_f, t_f; Y_i, t_i)\ . \tag{3.2}$$

Thus with

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{pmatrix}\ , \tag{3.3}$$

it follows that

$$\begin{aligned} H_0 &= \frac{1}{2}\hat{\pi}_a{}^b \hat{\pi}_b{}^a \\ &= -\frac{1}{2}\left[\left(2Y_{11}\frac{\partial}{\partial Y_{11}} + Y_{12}\frac{\partial}{\partial Y_{12}}\right)^2 + \left(2Y_{12}\frac{\partial}{\partial Y_{11}} + Y_{22}\frac{\partial}{\partial Y_{12}}\right)\left(Y_{11}\frac{\partial}{\partial Y_{12}} + 2Y_{12}\frac{\partial}{\partial Y_{22}}\right)\right. \\ &\quad \left. + \left(Y_{11}\frac{\partial}{\partial Y_{12}} + 2Y_{12}\frac{\partial}{\partial Y_{22}}\right)\left(2Y_{12}\frac{\partial}{\partial Y_{11}} + Y_{22}\frac{\partial}{\partial Y_{12}}\right) + \left(2Y_{22}\frac{\partial}{\partial Y_{22}} + Y_{12}\frac{\partial}{\partial Y_{12}}\right)^2\right]\ . \end{aligned} \tag{3.4}$$

Thus since the eigenfunctions of the Laplacian are

$$p_{\mathbf{s}}(Y) = Y_{11}^{s_1}(Y_{11}Y_{22} - Y_{12}^2)^{s_2} \tag{3.5}$$

it follows that the spectrum of the Hamiltonian specified by

$$H_0 p_{\mathbf{s}}(Y) = \lambda_{H_0}(\mathbf{s}) p_{\mathbf{s}}(Y) \tag{3.6}$$

is

$$\lambda_{H_0}(\mathbf{s}) = -2s_1^2 - 4s_2^2 - 4s_1 s_2 - s_1\ . \tag{3.7}$$

Thus the propagator is given by

$$G_0(Y,t;Y_1,t_1) = \int d\mathbf{s} \int d\bar{k} |c(\mathbf{s})|^{-2} p_{\mathbf{s}}(Y[k]) \overline{p_{\mathbf{s}}(Y_1[k])}\ e^{-i\lambda_{H_0}(\mathbf{s})(t-t_1)} \tag{3.8}$$

The Harish-Chandra $c$-function evaluated along the relevant contour is

$$|c(-\frac{1}{2}+iu_1, \frac{1}{4}+iu_2)|^{-2} = \pi u_1 \tanh \pi u_1 \ . \tag{3.9}$$

Now, given the form of $d\mu_{Helg}$

$$\begin{aligned} 0 &= \int d\mu_{Helg}\ \frac{\partial}{\partial u_2} \Big( p_{\mathbf{s}}(Y[k]) \overline{p_{\mathbf{s}}(Y_1[k])}\ e^{-i\lambda_L(\mathbf{s})(t-t_1)} \Big) \\ &= -i \int d\mu_{Helg}\ \Big( \ln\det(Y_1) - \ln\det(Y) - 4i(t-t_1)(s_1+2s_2) \Big) \\ &\quad \times p_{\mathbf{s}}(Y[k]) \overline{p_{\mathbf{s}}(Y_1[k])}\ e^{-i\lambda_L(\mathbf{s})(t-t_1)} \ . \end{aligned} \tag{3.10}$$

But along the contour used in the inverse Helgason transform,

$$s_2\ \overline{p_{\mathbf{s}}(Y_i[k])} = (-\frac{\partial}{\partial \ln\det(Y_i)} + \frac{1}{2}) \overline{p_{\mathbf{s}}(Y_i[k])}\ , \tag{3.11}$$

and

$$\begin{aligned} s_1 \overline{p_{\mathbf{s}}(Y_i[k])} = -((Y_i)_{11} \frac{\partial}{\partial (Y_i)_{11}} &+ (Y_i)_{22} \frac{\partial}{\partial (Y_i)_{22}} + (Y_i)_{12} \frac{\partial}{\partial (Y_i)_{12}} \\ &+ 1 - 2\frac{\partial}{\partial \ln\det(Y_i)}) \overline{p_{\mathbf{s}}(Y_i[k])}\ , \end{aligned} \tag{3.12}$$

hence

$$\begin{aligned} \ln\det(Y)\ G_0(Y,t;Y_i,t_i) = \Big( \ln\det(Y_i) + 4i(t-t_i)((Y_i)_{11} \frac{\partial}{\partial (Y_i)_{11}} + (Y_i)_{22} \frac{\partial}{\partial (Y_i)_{22}} + \\ (Y_i)_{12} \frac{\partial}{\partial (Y_i)_{12}}) \Big)\ G_0(Y,t;Y_i,t_i) \ . \end{aligned} \tag{3.13}$$

This allows us to do the $Y$ integration using the fact that

$$\int d\mu_{inv}(Y)\ G_0(Y_f,t_f;Y,t) G_0(Y,t;Y_i,t_i) = G_0(Y_f,t_f;Y_i,t_i) \ . \tag{3.14}$$

Hence we find that

$$G_1(Y_f,t_f;Y_i,t_i) = \lambda \int_{t_i}^{t_f} dt \Bigg[ (\ln\det(Y_i) + 4i(t-t_i)\bigg( (Y_i)_{11}\frac{\partial}{\partial (Y_i)_{11}} + (Y_i)_{22}\frac{\partial}{\partial (Y_i)_{22}} + \\ (Y_i)_{12}\frac{\partial}{\partial (Y_i)_{12}}\Bigg]\bigg) G_0(Y_f,t_f;Y_i,t_i)\ , \tag{3.15}$$

leading to

$$G_1(Y_f,t_f;Y_i,t_i) = \lambda \left( \frac{1}{2}(t_f-t_i)\big(\ln\det(Y_i) + \ln\det(Y_f)\big)\right) G_0(Y_f,t_f;Y_i,t_i)\ , \tag{3.16}$$

as in eqn (3.2).

It may be seen that the techniques used in this example allow the calculation of the perturbation expansion for a large class of interactions, leading to expressions of the form

$$G_n(Y_f,t_f;Y_i,t_i) = \lambda^n f_n(Y_f,t_f;Y_i,t_i)\ G_0(Y_f,t_f;Y_i,t_i)\ . \tag{3.17}$$

where $f_n$ is a multiplicative function.

As a last point in this section, we remark that, our expression for the first order term for the perturbation expansion for motion on positive symmetric $2\times 2$ matrices in the potential $V(Y) = \lambda \ln\det(Y)$ suggests the following Ansatz for the exact propagator for this system:

$$G_V(Y_f,t_f;Y_i,t_i) = G_0(Y_f,t_f;Y_i,t_i)e^{\left(\frac{1}{2}\lambda(t_f-t_i)(\ln\det(Y_i)+\ln\det(Y_f)) - \frac{1}{24}\lambda^2(t_f-t_i)^3\right)}\ . \tag{3.18}$$

That this is the correct exact expression may be checked by direct substitution into the relevant Schrödinger equation. We have thus added to the relatively small number of quantum systems whose propagator may be found in closed form. [2]

## 4. General Non-Compact $G/H$.

The same methods that were applied to quantum mechanics on positive symmetric matrices can be applied directly to the general case of non-compact $G/H$, at least for the trivially induced representation of $H$, and provided that $H$ is a maximal compact subgroup of $G$. The ingredients of this construction are a knowledge of the invariant Laplacian $L^{(2)}$,

[2] We believe that there is an error in Feynman and Hibbs [9] p64 equation (3-62) and that the correct last term in that equation should be $-\frac{f^2T^3}{24m}$ rather than $-\frac{fT^3}{24}$

its eigenfunctions, the Helgason transform and its inverse. These building blocks may be found in explicit detail in Helgason [11], Barut and Rączka [12] and Terras [10]. The result is essentially the same as equation 2.27, namely

$$\langle Y_f, t_f \mid Y_i, t_i \rangle = \int \left( \prod_{n=1}^{N} d\mu_{Helg}(\mathbf{s}_n, k_n) d\mu_{inv}(Y_n) \right) e^{\sum_{n=1}^{N} \ln(p_{\mathbf{s}}(Y_n[k])\overline{p_{\mathbf{s}}(Y_{n-1}[k])})} \\ e^{i\sum_{n=1}^{N}(\lambda_{H_0}(\mathbf{s}_n)+\lambda V(Y_n))\Delta t} \tag{4.1}$$

This is the transition amplitude between the initial state described by an element of $G/H$ given by $Y_i$, and a final state given by $Y_f$. As before, $Y[k]$ is $Y \in G/H$ conjugated by $k \in H$, and $p_{\mathbf{s}}(Y)$ are the power functions associated with the Laplacian $L^{(2)}$, which is taken to be a multiple of the free Hamiltonian $H_0$ with spectrum $\lambda_{H_0}$. In the path integral itself, $d\mu_{inv}$ is the canonical invariant measure on $G/H$, that is the volume form derived from the same metric that determines $L^{(2)}$. $d\mu_{Helg}$ is the measure for the inverse Helgason transform. This expression is quite complicated, but Gindikin and Karpelevic [13] have given an explicit form for the measure for such $G/H$. We do not record the specific forms here, but refer to original literature on this topic.

## 5. Conclusion

We have constructed a path integral for quantum mechanics on the space of positive symmetric matrices whose natural perturbation theory respects the global structure of the system. A more usual approach to the problem would be to use the phase-space or configuration space path integral. In the former case, one has (schematically)

$$G = \int d\mu_{Liouville} e^{i\int\ dt\ (p\dot{q}-H)}\ , \tag{5.1}$$

and it can be shown [2] that this expression satisfies the Schrödinger equation, so it must be equal to our expression. However, although the above equation is correct it is not clear how to use it to develop the sort of expression we are interested in.

An alternative way to treat many non-linear systems using a phase-space path integral is to follow Faddeev [14] and embed such a system in a vector space (for which our expression becomes the usual one) and use the constraints to restrict integration to the submanifold of interest. Unfortunately, this avenue is not open to us in the class of examples we have treated here since $Symm^+(n, \mathbf{R})$, rather than being defined by an equality (eg. a 2-sphere is defined by $x^2 + y^2 + z^2 = 1$ in $\mathbf{R}^3$) is defined by inequalities (eg. the one dimensional case is just the positive real line $x > 0$).

A further way one might have tried to treat the system dealt with in section three would be to realise that every positive symmetric matrix is the exponential of an arbitrary symmetric matrix, and that the exponential map is one-to-one and onto, so that we could have equivalently treated the space of arbitrary symmetric matrices which is a vector space. Apart from the fact that in a field theoretic version of this problem such a transformation would be extremely delicate (even $e^{\phi(x)}$ is ill-defined for usual scalar field theory), this isomorphism is not helpful in the case at hand. The reason is that, although the space of $n \times n$ symmetric matrices is isomorphic to $\mathbf{R}^{\frac{1}{2}n(n+1)}$ topologically, the natural metric on the former (which is the one we are using) has constant negative curvature. Furthermore the Hamiltonian on $\mathbf{R}^{\frac{1}{2}n(n+1)}$ in which we are interested is the one invariant under $GL(n, \mathbf{R})$ and so rather than the usual Laplacian $-\sum \frac{d^2}{dx_i^2}$ on $\mathbf{R}^{\frac{1}{2}n(n+1)}$, we need a quantisation adapted to this Hamiltonian. In other words, in order to treat the Hamiltonian of interest, we have to use a "strange" quantisation of $\mathbf{R}^{\frac{1}{2}n(n+1)}$ using $Symm(n, \mathbf{R}) \widetilde{\times} GL(n, \mathbf{R})$ rather than the Heisenberg-Weyl group. In fact, viewed as a theory on $\mathbf{R}^{\frac{1}{2}n(n+1)}$, our system for the free Hamiltonian is related to one of the integrable quantum systems discussed for example by Olshanetsky and Perelomov[15].

Lastly we would like to put forward a path integral for a field theory taking its values in positive symmetric matrices. By analogy with the quantum mechanical case, we suggest that the relevant expression should be

$$G = \prod_x \int \left(d\mu_{Helg}(\mathbf{s}(x), k(x)) d\mu_{inv}(Y(x))\right) e^{\int dx \ln \left( p_{\mathbf{s}}(Y(x)[k(x)]) \overline{p_{\mathbf{s}}(Y(x)[k(x)])} \right)} \\ e^{i \int dx (\lambda_L(\mathbf{s}(x)) + V(Y(x)))} \quad , \qquad (5.2)$$

and that perturbation theory based on this expression will respect the global structure of the configuration space.

An analog of formula 5.2 is an alternative solution to the problem of finding the phase space path integral for quantum gravity. This quiestion has been addressed in a large number of publications and often leads to measures of the form $d\mu_{Liouville}$ and hence to configuration space measures $d\mu_{invariant}$. It is then often argued that in dimensional regularization the form of the measure is unimportant and hence that finding the correct form of the measure is fruitless. As was clear from the example in section three, this is not the case for the path integral put forward here.

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**References**

[1] M. Stone, *Linear Tranformations in Hilbert Space,* Amer.Math.Soc.Colloq.Publ.15, (1932).
J. von Neumann, Ann. Math. **104**, (1931), 570.

[2] C. DeWitt-Morette, A. Maheshwari and B. Nelson, Phys. Reps. **50**(1979) 255.
J. Glimm and A. Jaffe *Quantum Physics - A Functional Integral Point of View*, Springer Verlag, New York (1981).
D.W. McLaughlin and L.S. Schulman, J. Math. Phys. **12**, (1971), 2520.
B.S. DeWitt, Rev. Mod. Phys. **29**, (1957), 377.
J.S. Dowker, J. Math. Phys. **17**, (1976), 1873.

[3] N. Linden and M.J. Perry, Nucl. Phys. **B357**, (1991), 289.

[4] M. Pilati, Phys. Rev. **D26**, (1982), 2645.
M. Pilati, Phys. Rev. **D28**, (1983), 729.
C.J. Isham, Proc. R. Soc. London **A351**, (1976), 209.

[5] N.P. Landsman and N. Linden, Nucl. Phys. **B365**, (1991), 121.

[6] C.J. Isham, in Relativity, Groups and Topology 2 (Les Houches 1983), B.S. DeWitt and R. Stora (eds.) North Holland, Amsterdam, (1984).

[7] B.S. DeWitt, Phys. Rev. **160**, (1967), 1113.
C.J. Isham and A.C. Kakas, Class. Quantum Grav. **1**, (1984), 621.

[8] J.E. Roberts, Commun. Math. Phys. **3**, (1966), 98.

[9] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* McGraw-Hill, New York, (1965).

[10] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications 1*, Springer Verlag, New York, (1985).
A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications 2*, Springer Verlag, New York, (1988).

[11] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York, (1984).

[12] A.O. Barut and R. Rączka, *Theory of Group Representations and Applications*, Polish Scientific Publishers, Warsaw, (1977).

[13] S. Gindikin and F. Karpelevic, Sov. Math. Dokl. **3**, (1962), 962.

[14] L.D. Faddeev, Teor. Matem. Fiz. **1**, (1969), 3.

[15] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. **94**, (1983), 313.